Math in Moscow, 2013-14 academic year
Ordinary differential equations
Midterm (March 24, 2014)
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1 (5). Sketch the integral curves of the following differential equation:

$$\dot{x} = (x^2 - 5x + 6)e^x$$

2 (10). The growth rate of the population is proportional to the size of the population. In the initial moment, the size of the population is 100. After 10 years, the size is 400. What is the size of the population after 15 years from the initial moment?

3 $(10 + 10)$. Find all solutions of the equation:

(a) $2xy' + y^2 = 1$; (b) $xy' = y - xe^{y/x}$;

4 $(5 + 5)$. Consider the following system:

$$\begin{cases} \dot{x} = \cos^2 x; \\ \dot{y} = \cos^2 y. \end{cases}$$

(a) Find all singular points of the system.

(b) Sketch its phase portrait in the domain $0 < x < 3\pi,\ 0 < y < 3\pi$. (Exact solutions are not needed.) Do not forget to show the direction of motion with arrows!

5 $(5 + 5 + 5)$. Consider system

$$\dot{x} = x^2, \quad \dot{y} = -y.$$

(a) Find a solution of this system with initial condition $x(0) = x_0$, $y(0) = y_0$ for arbitrary point (x_0, y_0).

(b) Draw phase portrait.

(c) Find all initial conditions (x_0, y_0) such that

$$\lim_{t\to-\infty} (x(t), y(t)) = (0, 0).$$

6 $(15 + 20)$. Find the equation of phase curves of the system satisfiying given initial condition.

(a) $\begin{cases} \dot{x} = (x - y + 1), \\ \dot{y} = 1/(x - y + 1), \end{cases}$ $x(0) = -2,\ y(0) = 0.$

(b) $\begin{cases} \dot{y} = y^5 + 3x^2 \cos y, \\ \dot{x} = x^3 \sin y - 3y^2 - 5y^4 x, \end{cases}$ $x(0) = 1,\ y(0) = 0.$

7 (20). Find a function $f(x, y)$ such that $H(x, y) = \sin(x^2 + y^2)$ be the first integral of system

$$\dot{x} = f(x, y), \quad \dot{y} = yx^2.$$

8 (25). Consider system

$$\dot{x} = (\alpha + 1)x, \quad \dot{y} = \alpha y, \quad \dot{z} = (\alpha - 1)z$$

For which values of parameter α there exists globally defined continous nonconstant first integral? Find (at least one) globally defined first integral for those values of α for which it exists.